- Scale analysis relevant to the tropics [large-scale synoptic systems]*

Goal: Use understanding of physically-relevant scales to reduce the complexity of the governing equations

**Reminder: Midlatitude scale analysis in dynamics*

Momentum Equations in Spherical Coordinates

$$\boxed{\frac{du}{dt} - \frac{uv\tan\lambda}{R_e} + \frac{uw}{R_e}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}} \boxed{+ 2\Omega v\sin\lambda - 2\Omega w\cos\lambda} \boxed{+ F_{rx}}$$

$$\boxed{\frac{dv}{dt} + \frac{u^2\tan\lambda}{R_e} + \frac{vw}{R_e}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y}} \boxed{- 2\Omega u\sin\lambda} \boxed{+ F_{ry}}$$

$$\boxed{\frac{dw}{dt} - \frac{u^2 + v^2}{R_e}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z}} \boxed{+ 2\Omega u\cos\lambda} \boxed{- g} \boxed{+ F_{rz}}$$

total derivative   pressure gradient   Coriolis   gravity   friction

To what extent are these terms important?

Scale Analysis

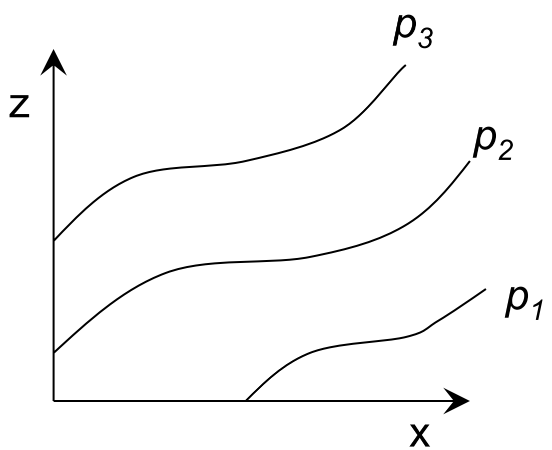
[*following Holton §11.2*]

- Goal: To determine relative importance of the terms in the basic equations for particular scales of motion.
- Approach: Estimate the following quantities
  1) The magnitude of the field variables.
  2) The amplitudes of fluctuations in the field variables. [*To estimate derivatives.*]
  3) The characteristic length, depth and time scales on which these fluctuations occur.

But first a coordinate change…

Vertical coordinate transformation

It is often convenient to replace height *z* by pressure *p*. Consider *p=p(x,z)*, where *p(x,z)* is assumed to be monotonic in *z* and *x* represents any horizontal coordinate, and the system of level curves sketched below:



Along each level curve, *p(x,z) = constant*. So let's consider the differential of *p*:

$$dp = 0 = \left.\frac{\partial p}{\partial x}\right|_z dx + \left.\frac{\partial p}{\partial z}\right|_x dz \Rightarrow \left.\frac{\partial z}{\partial x}\right|_p = -\frac{\left.\partial p / \partial x\right|_z}{\left.\partial p / \partial z\right|_x}$$

Under hydrostatic equilibrium: $\left.\frac{\partial z}{\partial x}\right|_p = \frac{\left.\partial p / \partial x\right|_z}{\rho g}$

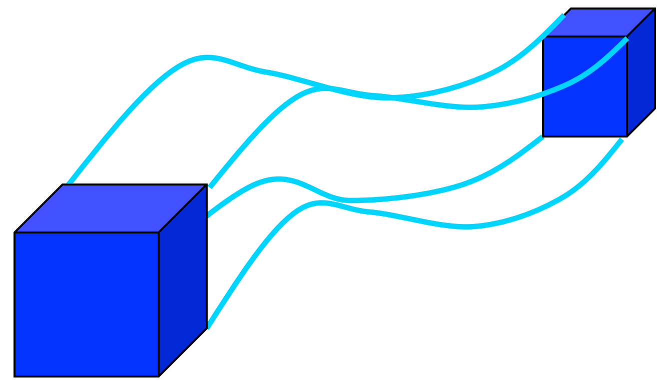
For further convenience, we introduce the geopotential: $\Phi = g \int_{z_0}^{z} dz$

Thus: $\left.\nabla_h \Phi\right|_p = \rho^{-1} \left.\nabla_h p\right|_z$

The notation ∇_h denotes horizontal components of the gradient operator.

Conservation of mass in pressure coordinate

Consider a parcel of mass δm which is conserved following the motion of the flow:



$$\frac{1}{\delta m}\frac{d(\delta m)}{dt} = 0$$

Assuming a density ρ, $\delta m = \rho\delta V$, and:

$$\frac{1}{\rho\delta V}\frac{d(\rho\delta V)}{dt} = 0 = \frac{g}{(\delta x\delta y\delta p)}\frac{d}{dt}\left(\frac{\delta x\delta y\delta p}{g}\right)$$

Pressure velocity

$$\omega \equiv \frac{dp}{dt}$$

$$\Rightarrow 0 = \left[\left(\frac{1}{\delta x}\frac{d\delta x}{dt} + \frac{1}{\delta y}\frac{d\delta y}{dt} + \frac{1}{\delta p}\frac{d\delta p}{dt}\right)\right] = \left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p}\right)$$

$$\Rightarrow \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p}\right) = 0$$

No density, hence no explicit time-dependence appearing in the mass conservation relationship expressed in pressure in the vertical.

“log pressure” coordinates

For the purposes of scale analysis, we’ll consider the natural logarithm of pressure rather than pressure itself:

$$z^* = -H \ln\left(\frac{p}{p_s}\right)$$

where p_s is a standard reference pressure and H is a scale height defined as:

$$H \equiv \frac{R_d T_s}{g}$$

In this coordinate:

$$w^* = \frac{dz^*}{dt} = -H \frac{d}{dt} \ln\left(\frac{p}{p_s}\right) = -\frac{H\omega}{p}$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_h \cdot \nabla + w^* \frac{\partial}{\partial z^*}$$

$$\frac{\partial \omega}{\partial p} = \frac{\partial w^*}{\partial z^*} - \frac{w^*}{H}$$

For an isothermal atmosphere at temperature Ts [~global surface temperature], z=z.*

However, for realistic temperature profiles in the troposphere, the difference between z and z is usually small.*

Scaling quantities

Scale	Symbol	Value
Horizontal velocity	U	$10\ \mathrm{m\ s^{-1}}$
Horizontal length	L	10^6 m
Vertical depth	H	10^4 m
Time	$\tau \sim L/U$	10^5 s
Vertical velocity	W	TBD
Geopotential fluctuation	$\delta\Phi$	TBD
Temperature fluctuation	δT	TBD

Physical Parameters

$g \approx 10\ m\, s^{-2}$ gravity

$R_e \approx 10^7 m$ radius of earth

$\lambda_0 = ?$ *We'll consider both middle and tropical latitudes….*

$f = 2\Omega \sin\lambda$ Coriolis parameter

$\nu = 10^{-5}\ m^2\, s^{-1}$ viscosity

Continuity + Horizontal Equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w^*}{\partial z^*}-\frac{w^*}{H}=0 \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\le\frac{U}{L}$$

In the horizontal, for nondivergent flow, $\nabla_h \cdot v=0$.

$$\frac{W}{H} \qquad W\le\frac{UH}{L}$$

To perform the horizontal scaling, we'll consider the magnitudes of terms relative to the horizontal advective scale: $\frac{U^2}{L}$

$$\frac{du}{dt}-\frac{uv\tan\lambda}{R_e}+\frac{uw^*}{R_e}=-\frac{\partial\Phi}{\partial x}+2\Omega v\sin\lambda-2\Omega w^*\cos\lambda+F_{rx}$$

$$\frac{dv}{dt}+\frac{u^2\tan\lambda}{R_e}+\frac{vw^*}{R_e}=-\frac{\partial\Phi}{\partial y}-2\Omega u\sin\lambda \qquad +F_{ry}$$

Re is the Reynolds Number

$$1 \qquad \frac{L}{R_e}\sim 0.1 \qquad \frac{LW}{R_eU}\le\frac{H}{R_e}\sim 10^{-3} \qquad \frac{\delta\Phi}{U^2} \qquad \frac{fL}{U}=\mathrm{Ro}^{-1} \qquad \frac{LW\Omega\cos\lambda}{U^2}\le \frac{\Omega H}{U}\sim 0.1 \qquad \frac{\nu}{LU}=\mathrm{Re}^{-1}\sim 10^{-12}$$

Ro is the Rossby Number

Continuity + Horizontal Equations

$$\frac{du}{dt} - \frac{uv\tan\lambda}{R_e} + \frac{uw^*}{R_e} = -\frac{\partial\Phi}{\partial x} + 2\Omega v\sin\lambda - 2\Omega w^*\cos\lambda + F_{rx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\lambda}{R_e} + \frac{vw^*}{R_e} = -\frac{\partial\Phi}{\partial y} - 2\Omega u\sin\lambda \qquad + F_{ry}$$

$$1 \qquad \frac{L}{R_e} \sim 0.1 \qquad \frac{LW}{R_e U} \le \frac{H}{R_e} \sim 10^{-3} \qquad \frac{\delta\Phi}{U^2} \qquad \frac{fL}{U} = \mathrm{Ro}^{-1} \qquad \frac{LW\Omega\cos\lambda}{U^2} \le \frac{\Omega H}{U} \sim 0.1 \qquad \frac{\nu}{LU} = \mathrm{Re}^{-1} \sim 10^{-12}$$

For mid-latitudes (λ~45°), $f \sim 10^{-4}$, so Ro ~ 0.1. Thus, the Coriolis term (~10) must be balanced by the geopotential gradient term, implying:

$$\delta\Phi \sim fUL \quad [\sim 1000\ \mathrm{m^2 s^{-2}}]$$

At low [tropical] latitudes, $f \le 10^{-5}$, so Ro ≥1, so Coriolis may not balance the pressure gradient force. In fact, for Ro ≥10, we expect:

$$\delta\Phi \sim U^2 \quad [\sim 100\ \mathrm{m^2 s^{-2}}]$$

Thus, for synoptic disturbances in the tropics, geopoential perturbations are an order of magnitude smaller than for similar-sized systems in mid latitudes, which has several important consequences…

Vertical equation

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = -\frac{1}{\rho}\left(\frac{\partial \Phi}{\partial p}\right)^{-1}\frac{\partial \Phi}{\partial z} = -\frac{g}{\rho}\left(\frac{\partial \Phi}{\partial p}\right)^{-1} \qquad \frac{\partial \Phi}{\partial p} = \frac{\partial \Phi}{\partial z^*}\frac{\partial z^*}{\partial p} = -\frac{H}{p}\frac{\partial \Phi}{\partial z^*}$$

$$\therefore -\frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{g}{H}\frac{p}{\rho}\left(\frac{\partial \Phi}{\partial z^*}\right)^{-1} = \frac{gR_dT}{H}\left(\frac{\partial \Phi}{\partial z^*}\right)^{-1}$$

$$\frac{dw^*}{dt} - \frac{u^2+v^2}{R_e} = \frac{gR_dT}{H}\left(\frac{\partial \Phi}{\partial z^*}\right)^{-1} + 2\Omega u\cos\lambda - g + F_{rz}$$

$$\frac{WU}{L} \le \frac{U^2H}{L^2} \sim 10^{-4} \qquad \frac{U^2}{R_e} \sim 10^{-5} \qquad \frac{gR_dT}{H}\left(\frac{\partial \Phi}{\partial z^*}\right)^{-1} \qquad U\Omega\cos\lambda \le 10^{-3} \qquad 10 \qquad \frac{\nu W}{H^2} \le \frac{\nu U}{HL} \sim 10^{-14}$$

The above scaling implies hydrostatic equilibrium: $\frac{\partial \Phi}{\partial z^*} = \frac{R_dT}{H}$

But it also provides a scaling for horizontal temperature perturbations*: $\delta T = \frac{H}{R_d}\frac{\partial}{\partial z^*}\delta\Phi \sim \frac{\delta\Phi}{R_d} \sim \frac{U^2}{R_d} \sim 0.3\ \mathrm{K}$

It is assumed that the geopotential consists of a mean component, which is a function of the vertical coordinate only, and a synoptic perturbation; since we're evaluating synoptic scale motion, the scaling applies to the perturbation part.

(Dry) Thermodynamic equation

$$\left(\frac{\partial}{\partial t}+v_h\cdot\nabla_h\right)T+\frac{w*HN^2}{R_d}=\frac{J}{c_P}$$

For δT~0.3 K, the first term on the left-hand side is of order 0.3K x 10^{-5} s^{-1} ~ 0.3K day^{-1}. When precipitation is absent, the *diabatic heating* term on the right-hand side comprises long-wave radiative emission, which is observed to be of order 1 K day^{-1}. Thus,

$$\frac{w*HN^2}{R_d}\approx\frac{J}{c_p}\Rightarrow w*\approx\frac{J}{c_p}\frac{R_d}{HN^2}$$

For the tropical tropopshere, N^{-1}~100 s, so:

$$w*\sim W\ \sim 0.3\,\mathrm{cms}^{-1}$$

In the absence of precipitation, synoptic scale tropical vertical velocities are much smaller than in similar-sized extratropical systems. Also, from the continuity equation, the divergence of horizontal wind is of order 10^{-7}; thus the flow is essentially nondivergent.

(Brief overview of) precipitating tropical synoptic systems

What modifications are anticipated for precipitating tropical synoptic systems? For precipitating tropical synoptic systems, rain rates of 20 mm day^{-1} are typical. For a unit area cross-section, this precipitation rate implies a daily condensation of 20 kg of $H_2 0$. [Think density x area, which is dimensionally mass/length.] Since L_c~2.5 x 10^6 J kg^{-1}, the net condensational energy input per day into a unit area atmospheric column is:

$$\frac{d}{dt}\left(m_{H_2O}L_c\right) \approx 5\times10^7 \text{ Jm}^{-2}\text{day}^{-1}$$

Assuming this heating is spread uniformly over the entire tropospheric depth, or over a unit area column of tropospheric air mass, implies a daily heating rate per unit air mass, i.e., $(p_s - p_t)g^{-1}$, of:

$$Q_{cnv} = \frac{J_{cnv}}{c_p} = \frac{{}^{d}\!/\!{}_{dt}\left(m_{H_2O}L_c\right)g}{c_p(p_s - p_t)} \approx 5 \text{ Kday}^{-1}$$

As we'll see later, the distribution of condensational heating via tropical convection is nonuniformly distributed in the vertical and tends to maximize in the mid- to upper-troposphere (~300 to 400 mb), with heating rates of order 10 K day^{-1}. Recalling that the radiative heating rate is ~ 1 K day^{-1}, the above implies a vertical velocity scale ~ 10x larger than in nonprecipitating regions. This in turn implies a relatively large component of the divergent flow.